Session04

Sunday, November 19, 2023 10:55 AM

$$\mathcal{L} = B\int_{t=0}^{\infty} e^{-\beta t}\,\frac{C(t)^{1-\theta}}{1-\theta}\,dt$$

$$+\lambda\left[k(0) + \int_{t=0}^{\infty} e^{-R(t)}\,w(t)\,e^{(n+g)t}\,dt - \int_{t=0}^{\infty} e^{-R(t)}\,c(t)\,e^{(n+g)t}\,dt\right]$$

$$B\,e^{-\beta t}\,\frac{1-\theta}{1-\theta}\,c(t)^{-\theta} - \lambda\,e^{-R(t)}\,e^{(n+g)t} = 0$$

$$\mathcal{L} = U(x,y) + \lambda(I - P_x x - P_y y)$$

$$U'_x - \lambda P_x = 0 \;\Rightarrow\; U'_x = \lambda P_x \qquad \frac{U'_x}{U'_y} = \frac{P_x}{P_y}$$

$$U'_y - \lambda P_y = 0 \qquad U'_y = \lambda P_y$$

$$B\,e^{-\beta t}\,c(t)^{-\theta} = \lambda\,e^{-R(t)+(n+g)t}$$

$$\int_{\tau=0}^{t} r(\tau)\,d\tau$$

$$\ln B - \beta t - \theta \ln C(t) = \ln\lambda - R(t) + (n+g)t$$

$$-\beta - \theta\,\frac{\dot{c}(t)}{c(t)} = -r(t) + (n+g)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - n - g - \beta}{\theta}$$

معادله اویلر

$$\beta = \rho - (1-\theta)g - n$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \cancel{n} - \cancel{g} - \rho + (1-\theta)\cancel{g} + \cancel{n}}{\theta}$$

نرخ رشد مصرف سرانه موثر $$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{r(t)-\rho}{\theta} - g$$

نرخ رشد مصرف سرانه $$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho}{\theta} = 0$$

$$\bar{C} = C_t = C_{t+1} \quad \text{Steady State}$$

$$r = \rho$$

$$B\,e^{-\beta t}\,c(t)^{-\theta}\,\Delta C$$

مطلوبیت نهایی ناشی از افزایش مصرف به اندازه $\Delta C$

$t + \Delta t$ دوره بعد

$$B\,e^{-\beta(t+\Delta t)}\,\underbrace{c(t+\Delta t)}^{-\theta}\,\Delta c$$

$$C(t+\Delta t) = c(t)\,e^{\frac{\dot{c}(t)}{c(t)}\Delta t}$$

$$C(t+\Delta t) = C(t)\, e^{\overbrace{\frac{\dot c(t)}{c(t)}\Delta t}}$$

$$\beta e^{-\beta(t+\Delta t)} \; C(t)^{-\theta} \; e^{-\frac{\dot c(t)}{C(t)}\theta \Delta t} \; \Delta C \, e^{(r(t)-n-g)\Delta t}$$

$$\cancel{\beta} e^{-\beta t} \cancel{c(t)^{-\theta}} \cancel{\Delta C} = \cancel{\beta} e^{-\beta(t+\Delta t)} \cancel{c(t)^{-\theta}} \; e^{-\frac{\dot c(t)}{c(t)}\theta \Delta t} \; \cancel{\Delta C} \, e^{(r(t)-n-g)\Delta t}$$

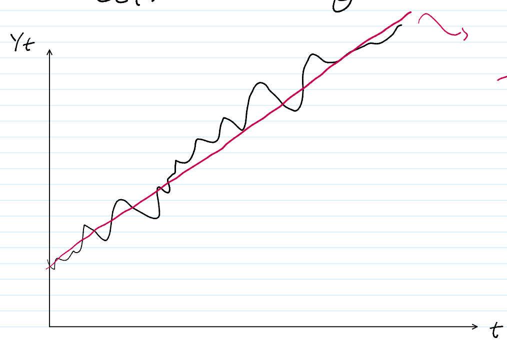
$$1 = e^{-\beta\Delta t} \; e^{-\frac{\dot c(t)}{c(t)}\theta\Delta t} \; e^{(r(t)-n-g)\Delta t}$$

از طرفین ln بگیریم:

$$0 = -\beta \cancel{\Delta t} - \frac{\dot c(t)}{c(t)}\theta \cancel{\Delta t} + (r(t)-n-g)\cancel{\Delta t}$$

$$\theta \frac{\dot c(t)}{c(t)} = r(t)-n-g-\beta$$

$$\frac{\dot c(t)}{c(t)} = \frac{r(t)-\rho-\theta g}{\theta}$$




روند
Trend

State variable

متغیر وضعیت

$\dot c = 0$

$$\frac{\dot c(t)}{c(t)} = \frac{f'(k(t))-\rho-\theta g}{\theta} = 0$$

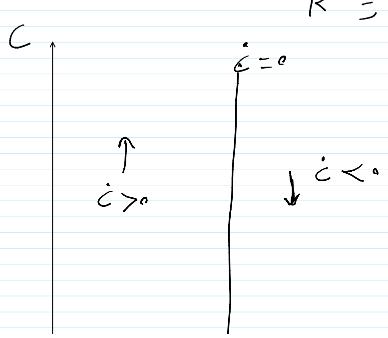
$$f'(k) = \rho + \theta g$$

$$f(k) = k^{\alpha}$$

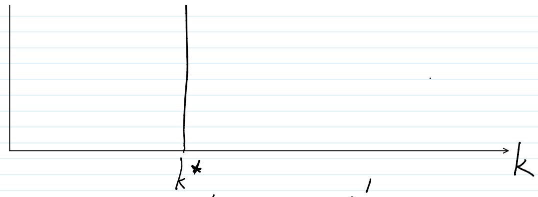
$$f'(k) = \alpha k^{\alpha-1} = \rho + \theta g$$

$$k^* = \left(\frac{\rho+\theta g}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

$$k^* = \left(\frac{\alpha}{\rho+\theta g}\right)^{\frac{1}{1-\alpha}}$$




پویایی c

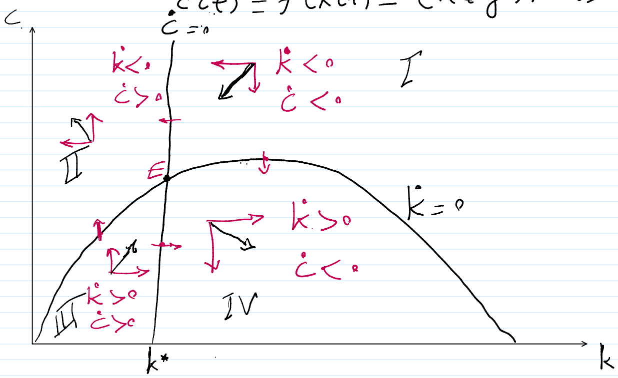


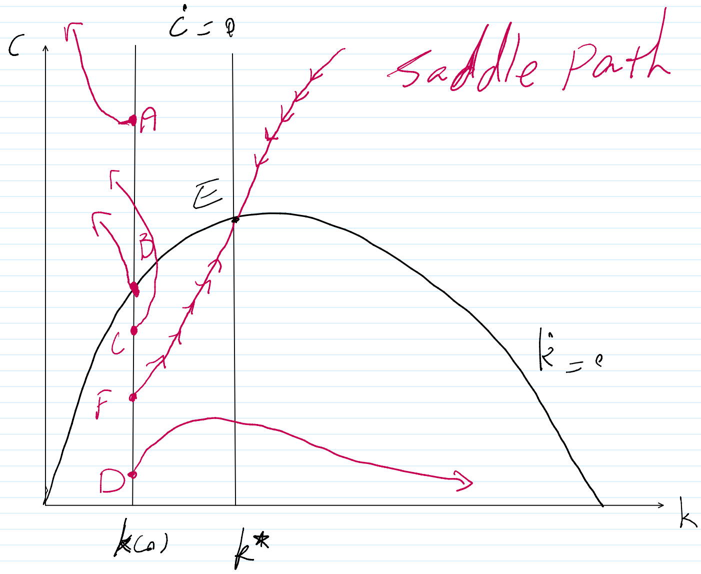

if $k < k^* \Rightarrow f'(k) > f'(k^*)$

k بیشتر

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t) = 0$$

$$c(t) = f(k(t)) - (n+g)k(t)$$




مسیر زینی

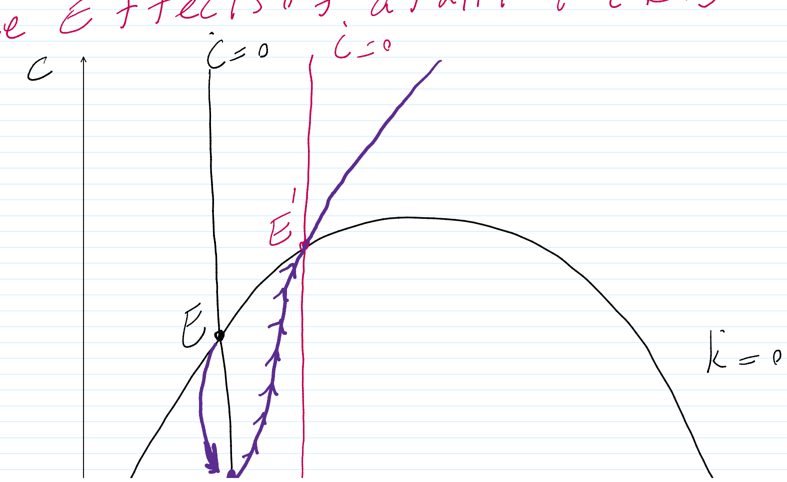
$r = \rho$

$f'(k^*) = \rho$

$f'(k^*_G) = n$

$k^*_G > k^*$

## The Effects of a Fall in the Discount Rate

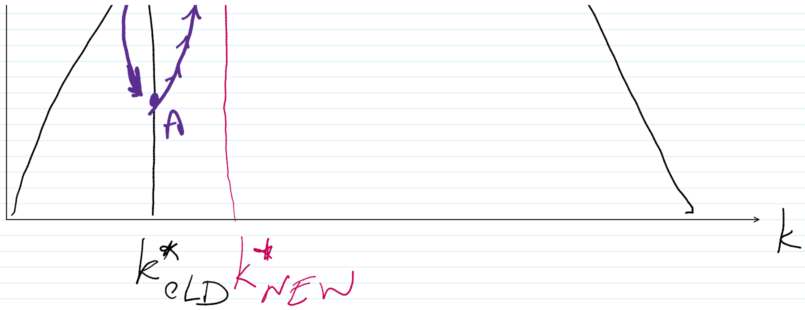



$f'(k^*_{OLD}) = \rho_0$

$f'(k^*_{NEW}) = \rho_1$

$\rho_1 < \rho_0$

$k^*_{NEW} > k^*_{OLD}$




$k^*_{NEW} > k^*_{OLD}$

# The Rate of Adjustment and the slope of Saddle Path

$$\dot c \simeq \frac{\partial \dot k}{\partial k}(k-k^*) + \frac{\partial \dot c}{\partial c}(c-c^*)$$

$$\dot k \simeq \frac{\partial \dot k}{\partial k}(k-k^*) + \frac{\partial \dot k}{\partial c}(c-c^*)$$

$$\tilde k \equiv k - k^* \qquad \tilde c \equiv c - c^*$$

$$\dot{\tilde c} = \dot c \qquad \dot{\tilde k} = \dot k$$

$$\dot{\tilde c} \simeq \frac{\partial \dot c}{\partial k}\tilde k + \frac{\partial \dot c}{\partial c}\tilde c$$

$$\dot{\tilde k} \simeq \frac{\partial \dot k}{\partial k}\tilde k + \frac{\partial \dot k}{\partial c}\tilde c$$

$$\frac{\dot c}{c} = \frac{f'(k^*) - \rho - \theta g}{\theta}$$

$$\dot c = \left[\frac{f'(k^*) - \rho - \theta g}{\theta}\right] c$$

$$\frac{\partial \dot c}{\partial c} = \frac{\partial \dot{\tilde c}}{\partial c} \qquad c^*, k^*$$

$$\frac{\partial \dot c}{\partial k} = \frac{\partial \dot{\tilde c}}{\partial k}$$

$\cdot \quad f'(k^*) \quad \rho \quad \theta g$

$\partial k \quad \partial k$

$$\frac{\partial \dot{c}}{\partial c} = \frac{f'(k^*) - \rho - \theta g}{\theta} = 0 \leftarrow$$

$$\frac{\partial \dot{c}}{\partial k} = \frac{f''(k^*)\, c^*}{\theta}$$

$$\dot{\tilde{c}} = \frac{f''(k^*)\, c^*}{\theta} \tilde{k}$$

$$\dot{k} = f(k) - c - (n+g)k$$

$$\frac{\partial \dot{k}}{\partial k} = f'(k^*) - (n+g)$$

$$\frac{\partial \dot{k}}{\partial c} = -1$$

$$\dot{\tilde{k}} \simeq [f'(k^*) - (n+g)]\tilde{k} - \tilde{c}$$

$$f'(k^*) = \rho + \theta g$$

$$\dot{\tilde{k}} \simeq [\underbrace{\rho + \theta g - (n+g)}_{\beta}]\tilde{k} - \tilde{c}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} \simeq \frac{\beta \tilde{k}}{\tilde{k}} - \frac{\tilde{c}}{\tilde{k}}$$

$$\dot{\tilde{c}} \sim f(k^*)\, c^*\, \tilde{k}$$

$$\frac{\dot{\tilde c}}{\tilde c} \simeq \frac{f''(k^*)\, c^*}{\theta} \frac{\tilde k}{\tilde c}$$

$$\frac{\dot{\tilde k}}{\tilde k} \simeq \beta - \frac{\tilde c}{\tilde k}$$

$$\frac{\dot{\tilde c}}{\tilde c} \simeq \frac{f''(k^*)\, c^*}{\theta} \frac{\tilde k}{\tilde c}$$

$$\dot{\tilde c} = \dot c$$

$$\frac{\dot{\tilde c}}{\tilde c} = \frac{\dot c}{c - c^*} \qquad \frac{\dot{\tilde c}}{\tilde c} = \frac{\dot{\tilde k}}{\tilde k}$$

$$\frac{\dot c}{c} \neq \frac{\dot c}{c - c^*}$$

$$\frac{\dot{\tilde c}}{\tilde c} = \mu \simeq \frac{f''(k^*)\, c^*}{\theta} \frac{\tilde k}{\tilde c}$$

$$\frac{\tilde c}{\tilde k} = \frac{f''(k^*)\, c^*}{\theta} \cdot \frac{1}{\mu}$$

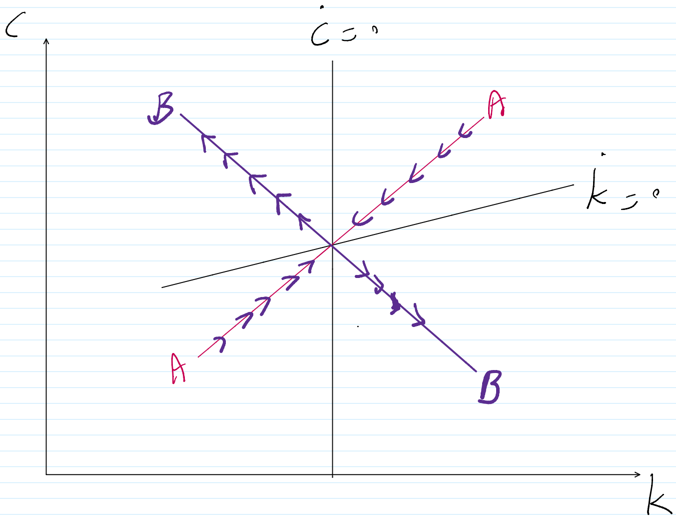
$$\mu \simeq \beta - \frac{\tilde c}{\tilde k}$$

$$\mu = \beta - \frac{f''(k^*)\, c^*}{\theta} \frac{1}{\mu}$$

$= \mu$

$$\mu^2 - \beta\mu + \frac{f''(k^*)c^*}{\theta} = 0$$

$$\mu = \frac{\beta \pm \left[\beta^2 - 4f''(k^*)k^*/\theta\right]^{1/2}}{2}$$




## The Effects of Government Purchase:

$$Y = C + I + G$$

$$\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$$

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} \left[w(t) - G(t)\right] e^{(n+g)t} d$$

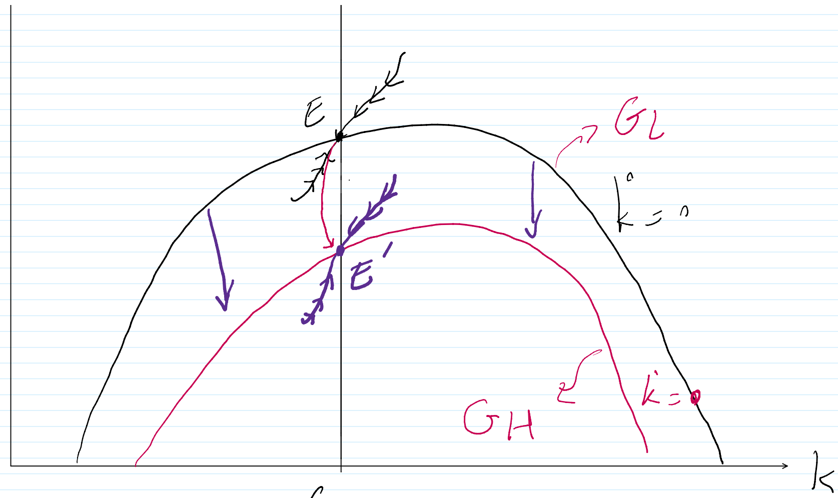
$$G(t) = T(t)$$

$$G_L \longrightarrow G_H \qquad G_H > G_L$$


افزایش دائمی مخارج
دولت از $G_L$ به $G_H$

l t

دولت از $G_L$ به $G_H$



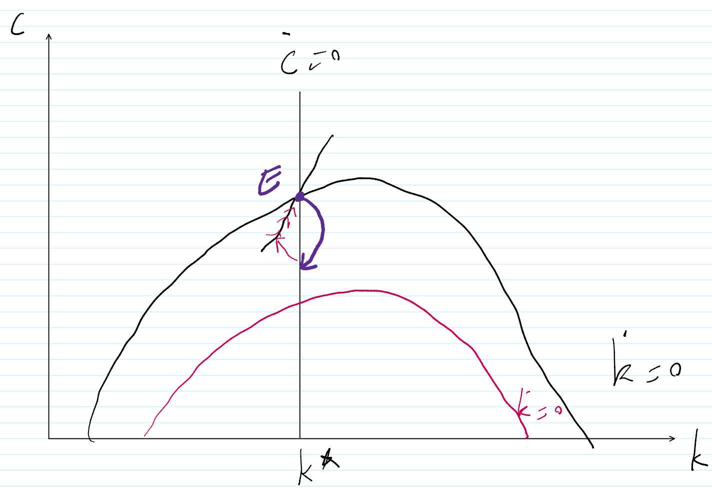

$$\dot{c}=0 \qquad f'(k^*)=\rho+\theta g$$

$$\dot{k}=0 \qquad f(k^*)-c^*-G^*-(n+g)k^*=0$$

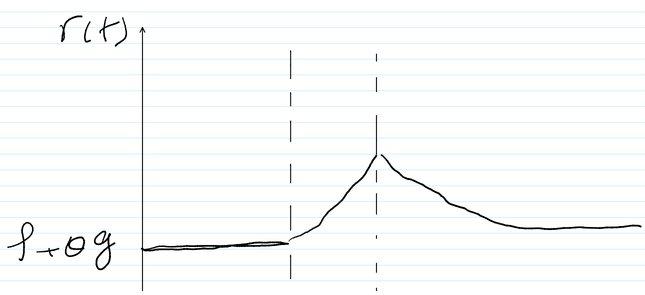
$$c^*=f(k^*)-G^*-(n+g)k^*$$

$$G_L^* \longrightarrow G_H^* \uparrow$$

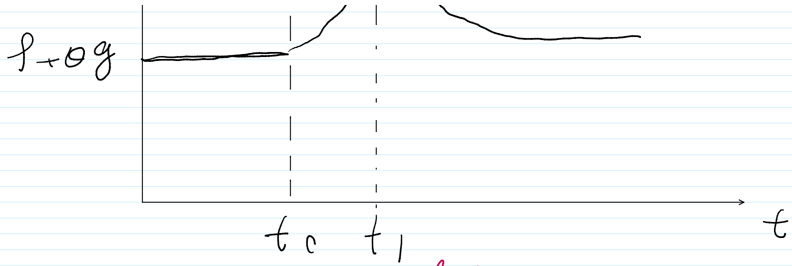
افزایش موقت $G_L$ به $G_H$




## Consumption Smoothing




$$r(t)=f'(k(t))$$




# The Diamond Model:

## Overlapping-generations

نسل‌های همپوشان

$c(t)$ پیوسته

$C_t$ گسسته

$$L_t = (1+n)L_{t-1}$$

$$c_{1t} \quad , \quad c_{2t}$$

old

$$U_t = \frac{c_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{c_{2t+1}^{1-\theta}}{1-\theta} \qquad \theta > 0 \, , \, \rho > -1$$

young

$$\frac{1}{1+\rho} > 1 \qquad 1+\rho < 1$$

$$\rho < 0$$

$$A_t = (1+g)A_{t-1}$$

$$Y_t = F(K_t , A_t L_t)$$

$$y_t = f(k_t)$$

$$r_t = f'(k_t)$$

$$w_t = f(k_t) - k f'(k_t)$$

$$K_{t+1} = w_t A_t L_t - c_{1t} L_t = (w_t A_t - c_{1t}) L_t$$

$$K_{t+1} = w_t A_t L_t - C_{1t} L_t = (w_t A_t - C_{1t}) L_t$$

$$S_t = I_t = K_{t+1} - K_t + \cancel{\delta K_t}$$

صفر

صفر

$$I_t = K_{t+1} = (w_t A_t - C_{1t}) L_t$$

مصرف سرانه

$$C_{1t}, \quad C_{2t}$$

$$K_{t+1} = (1-\delta) K_t + I_t$$

## Household Behavior:

$$C_{2t+1} = (1 + r_{t+1})(w_t A_t - C_{1t})$$

$$\frac{C_{2t+1}}{1 + r_{t+1}} = w_t A_t - C_{1t}$$

$$C_{1t} + \frac{C_{2t+1}}{1 + r_{t+1}} = w_t A_t$$

$$\text{Max} \quad U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

s.t

$$C_{1t} + \frac{C_{2t+1}}{1+r_{t+1}} = w_t A_t$$

$$\mathcal{L} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda \left[ A_t w_t - \left( C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} \right) \right]$$

$$\frac{\partial \mathcal{L}}{\partial C_{1t}} = C_{1t}^{-\theta} - \lambda = 0 \Rightarrow \lambda = C_{1t}^{-\theta}$$

$\partial c_{1t}$

$$\frac{\partial \mathcal{L}}{\partial c_{2t}} = \frac{1}{1+\rho} c_{2t+1}^{-\theta} - \frac{\lambda}{1+r_{t+1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = A_t w_t - \left( c_{1t} + \frac{1}{1+r_{t+1}} c_{2t+1} \right) = 0$$

$$\frac{c_{1t}^{-\theta}}{1+r_{t+1}} = \frac{1}{1+\rho} c_{2t+1}^{-\theta}$$

$$c_{1t}^{-\theta} = 1+r_{t+1} \left[ \frac{1}{1+\rho} c_{2t+1}^{-\theta} \right]$$

معادله اویلر

$$c_{1t} = \left( \frac{1+\rho}{1+r_{t+1}} \right)^{\frac{1}{\theta}} c_{2t+1}$$

$$c_{2t+1} = \left( \frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\theta}} c_{1t}$$

$$c_{1t} + \frac{1}{1+r_{t+1}} c_{2t+1} = w_t A_t$$

$$c_{1t} + \frac{1}{1+r_{t+1}} \left( \frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\theta}} c_{1t} = w_t A_t$$

$$c_{1t} + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} c_{1t} = w_t A_t$$

$$\star \quad \underbrace{\left[ \frac{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} \right]} c_{1t} = w_t A_t$$

$$\underbrace{(1+\rho)^{\frac{1}{\theta}}}_{\to Z}$$

# $S_t = w_t A_t - C_{1t} = w_t A_t - \frac{1}{Z} w_t A_t = \frac{Z-1}{Z} w$

$$Z C_{1t} = w_t A_t \Rightarrow C_{1t} = \frac{1}{Z} w_t A_t$$

$s(r)$ پس‌انداز

$$s(r) = \frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r)^{\frac{1-\theta}{\theta}}}$$

نرخ پس انداز

$$C_{1t} = \left(1 - s(r_{t+1})\right) A_t w_t$$

$$\Lambda = (1+r)^{\frac{1-\theta}{\theta}}$$

$$\frac{\partial s(r)}{\partial r}$$

$$s(r) = \frac{\Lambda}{(1+\rho)^{\frac{1}{\theta}} + \Lambda}$$

$$\frac{\partial s(r)}{\partial r} = \frac{\partial s(r)}{\partial \Lambda} \cdot \frac{\partial \Lambda}{\partial r} =$$

$$\frac{\partial s(r)}{\partial \Lambda} = \frac{(1+\rho)^{\frac{1}{\theta}} + \cancel{\Lambda} - \cancel{\Lambda}}{\left((1+\rho)^{\frac{1}{\theta}} + \Lambda\right)^2}$$

$$\frac{\partial \Lambda}{\partial r} = \frac{1-\theta}{\theta} (1+r)^{\frac{1-2\theta}{\theta}}$$

$$\Lambda = (1+r)^{\frac{1-\theta}{\theta}}$$

$$\frac{\partial s(r)}{\partial r} = \underbrace{\frac{\overbrace{(1+\rho)}^{+}{}^{\frac{1}{\theta}}}{\left[\underbrace{(1+\rho)^{\frac{1}{\theta}}}_{+} + \Lambda\right]^{2}}}_{+} \cdot \frac{1-\theta}{\theta} \underbrace{(1+r)^{\frac{1-}{}}}_{+}$$

$v_t A_t$

خس

$$\frac{2\theta}{\theta}$$

if $\theta < 1$ $\quad \dfrac{\partial S(r)}{\partial r} < 0$

if $\theta > 1$ $\quad \dfrac{\partial S(r)}{\partial r} > 0$

if $\theta = 1$ $\quad \dfrac{\partial S(r)}{\partial r} = 0$

## The Dynamics of Model:

$$K_{t+1} = w_t A_t L_t - C_{1t} L_t = (w_t A_t - C_{1t}) L_t$$

$$K_{t+1} = s(r_{t+1}) \cdot A_t w_t L_t$$

$$k_{t+1} = \frac{K_{t+1}}{A_{t+1} L_{t+1}} = \frac{s(r_{t+1}) A_t w_t L_t}{A_{t+1} L_{t+1}} = \frac{s(r_{t+1}) w_t}{\frac{A_{t+1}}{A_t} \frac{L_{t+1}}{L_t}}$$

$$k_{t+1} = \frac{s(r_{t+1}) w_t}{(1+g)(1+n)} = \frac{1}{(1+g)(1+n)} s(f'(k_{t+1})) \left[ f(k_t) - k_t \right.$$

$$k_{t+1} = k_t = k$$

$$\boxed{y = f(k_t) = k_t^{\alpha}} \Rightarrow f'(k_t) = \alpha k_t^{\alpha - 1}$$

$\theta = 1$

$$s(r) = \frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r)^{\frac{1-\theta}{\theta}}} = \frac{1}{1+\rho+1} = \frac{1}{2}$$

$f'(k_t)]$

$+ f$

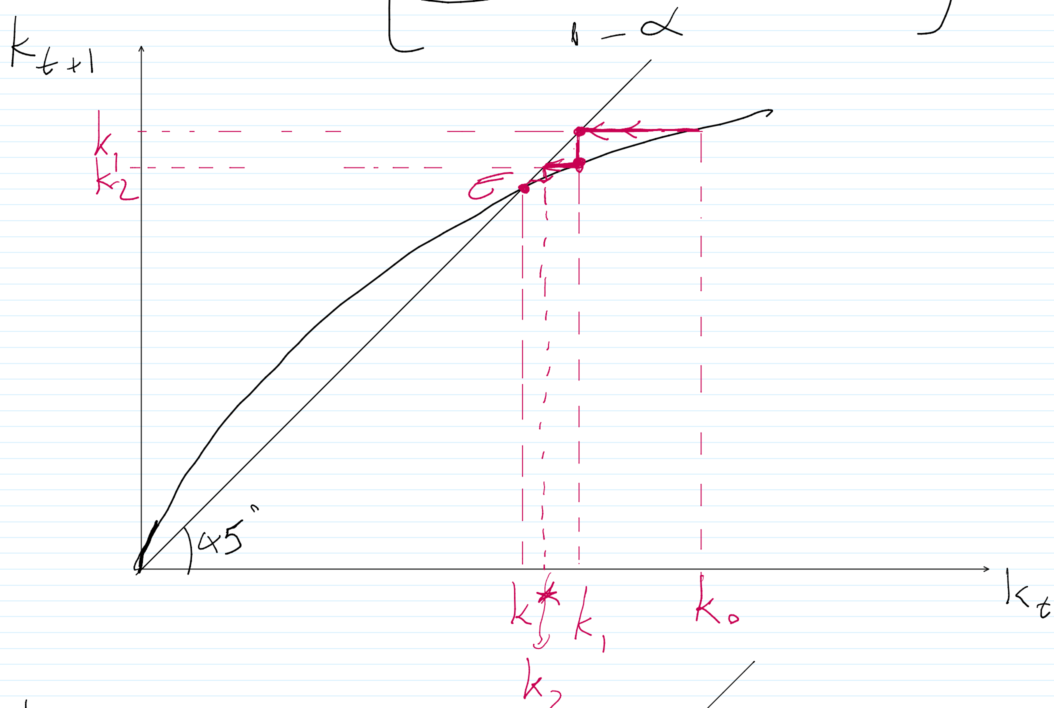
$$s(r) = \frac{}{(1+\rho)^{\frac{1}{\theta}} + (1+r)^{\frac{1-\theta}{\theta}}} = \frac{}{1+\rho+1} \quad 2$$

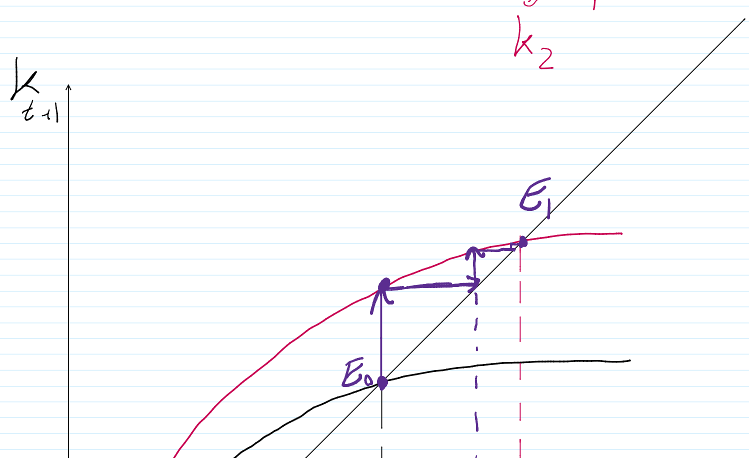
$$k_{t+1} = \frac{1}{(1+g)(1+n)} \frac{1}{2+\rho} \left[ k_t^{\alpha} - k_t \underbrace{\alpha k_t^{\alpha-1}}_{\alpha k_t^{\alpha}} \right.$$

$$\boxed{k_{t+1} = \frac{(1-\alpha)}{(1+g)(1+n)(2+\rho)} k_t^{\alpha}}$$

$$k^* = \underbrace{k_{t+1} = k_t}  \Rightarrow \quad k^{*1-\alpha} = \frac{(1+g)(1+n)(2+\rho)}{1-\alpha}$$

$$k^* = \left[ \frac{(1+g)(1+n)(2+\rho)}{1-\alpha} \right]^{\frac{1}{1-\alpha}}$$



$+ f$


## The Speed of Convergence:

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)} k_t^{\alpha}$$

$$k^* = \frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)} {k^*}^{\alpha}$$

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

$$y^* = \left(\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right)^{\frac{\alpha}{1-\alpha}}$$

$$k_{t+1} \simeq k^* + \left(\left.\frac{dk_{t+1}}{dk_t}\right|_{k=k^*}\right)(k_t - k^*)$$

$$k_{t+1} - k^* \simeq \lambda (k_t - k^*)$$

سرعت همگرایی

$$k_{t+1} - k^* \simeq \lambda^t (k_0 - k^*)$$

$$\lambda = \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t=k^*} = \frac{\alpha(1-\alpha)}{(1+n)(1+g)(2+\rho)} {k^*}^{\alpha-1}$$

$$\lambda = \alpha \cancel{\frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)}} \cancel{\left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]}$$

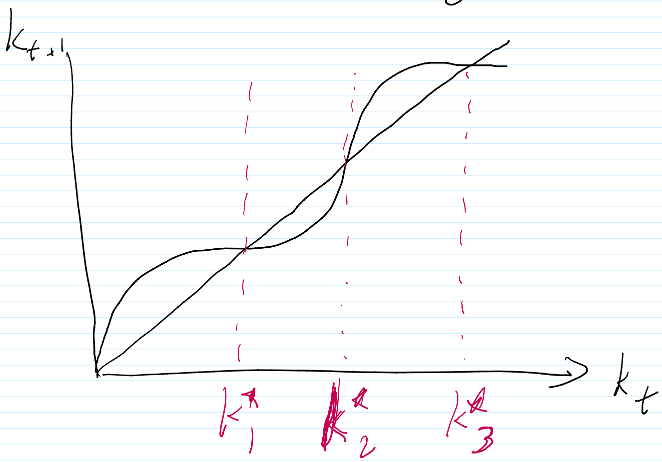
$$\lambda = \alpha$$

$$\frac{\cancel{\alpha} - 1}{1 - \alpha}$$

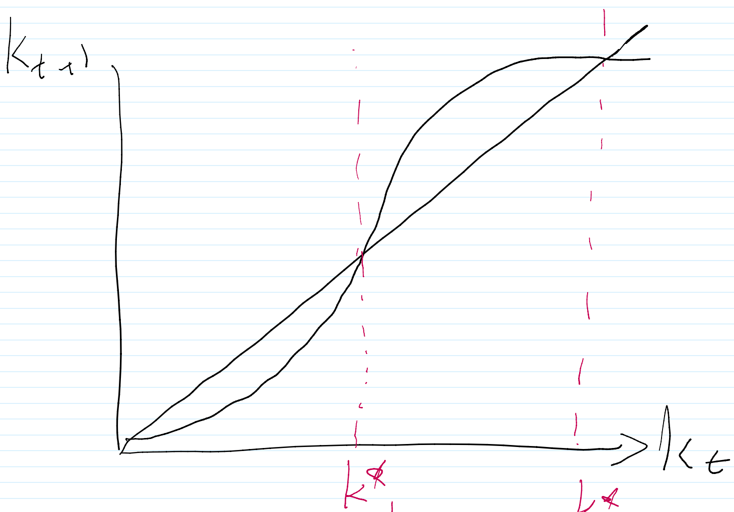
# The General Case:

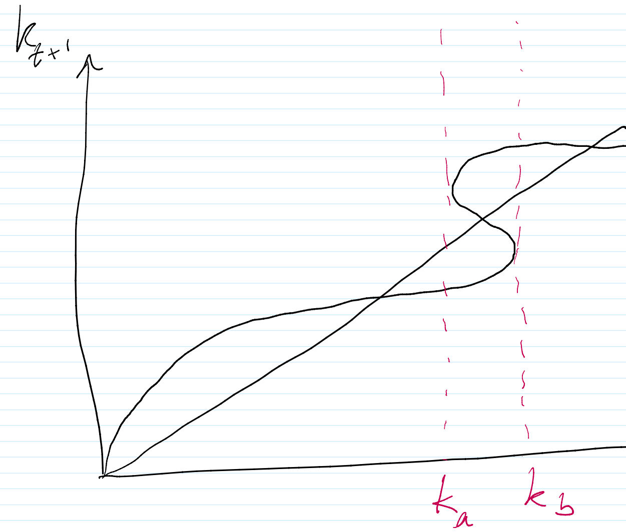
$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))$$

$$\frac{f(k_t) - k_t f'(k_t)}{f(k_t)} \quad f(k_t$$







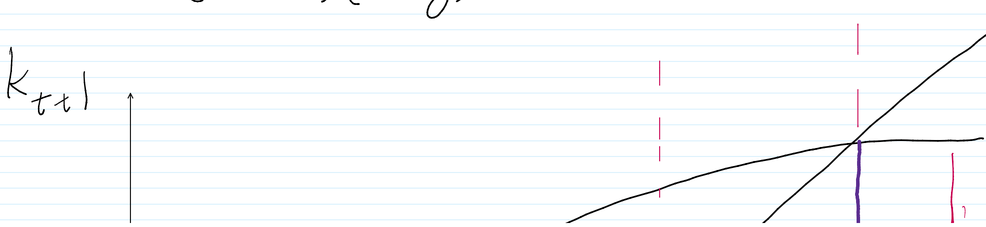




# Government in the Diamond Model:

$G_t = T_t$

$$(1-\alpha)k_t^\alpha - G_t$$

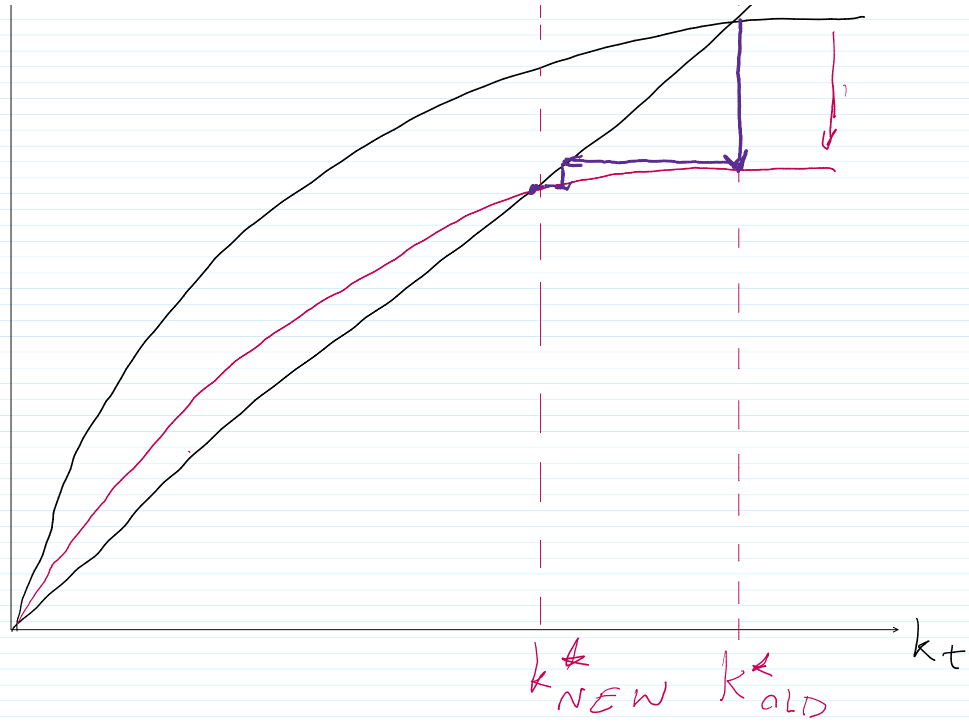
$$k_{t+1} = \frac{1}{(1+n)(1+g)(2+\rho)}\left[(1-\alpha)k_t^\alpha - G_t\right]$$



)

$\to k_t$

$\longrightarrow k_t$



$k_t$
$k^*_{NEW}$
$k^*_{OLD}$